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THE LIMIT-POINT, LIMIT-CIRCLE NATURE OF RAPIDLY OSCILLATING POT--ETC(U)

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MRC-TSR-1676

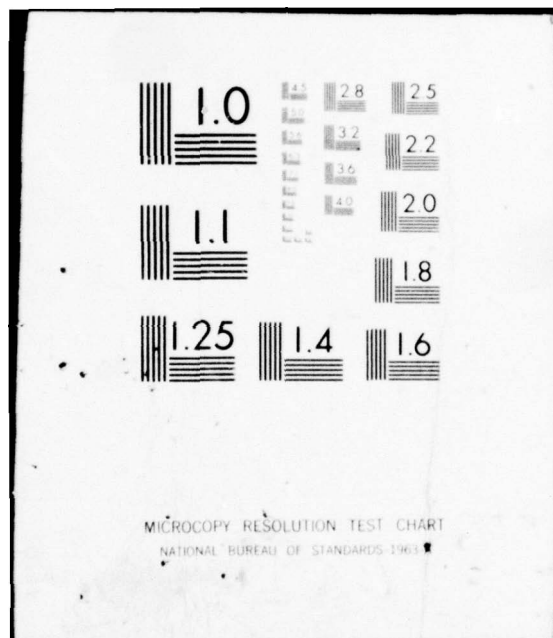
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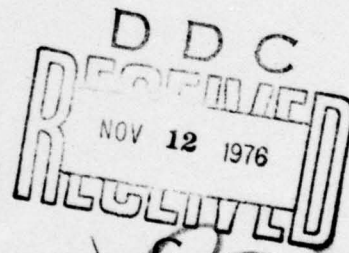
THE LIMIT-POINT, LIMIT-CIRCLE
NATURE OF RAPIDLY OSCILLATING
POTENTIALS

F. V. Atkinson, M. S. P. Eastham,
and J. B. McLeod

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September 1976

(Received September 25, 1975)



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ABSTRACT

The report analyses the Weyl limit-point, limit-circle classification,
i.e. the number of linearly independent solutions in $L^2(0, \infty)$, of the
equation

$$y''(x) - q(x)y(x) = 0 \quad (0 \leq x < \infty),$$

where $q(x)$ has the form

$$q(x) = x^\alpha p(x^\beta),$$

α and β being positive constants and $p(t)$ a real continuous periodic
function of t .

AMS (MOS) Subject Classification: 34B20

Key Words: Ordinary differential operators
Boundary-value problems
Self-adjointness
Asymptotics

Work Unit Number 1 (Applied Analysis)

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THE LIMIT-POINT, LIMIT-CIRCLE NATURE OF RAPIDLY
OSCILLATING POTENTIALS

F. V. Atkinson, M. S. P. Eastham, and J. B. McLeod

1. Introduction

We consider the Weyl limit-point, limit-circle classification, i.e. the number of linearly independent solutions in $L^2(0, \infty)$, of the second-order equation

$$y''(x) - q(x)y(x) = 0 \quad (0 \leq x < \infty), \quad (1.1)$$

where the real-valued potential $q(x)$ has the form

$$q(x) = x^\alpha p(x^\beta). \quad (1.2)$$

Here α and β are positive constants and $p(t)$ is a continuous periodic function of t . We denote by a the period of $p(t)$.

It is perhaps worth remarking briefly on the significance of the classification into limit-point and limit-circle for general real-valued potentials $q(x)$. In the limit-point case, the linear operator

$$-\frac{d^2}{dx^2} + q(x)$$

associated with the equation (1.1) and some homogeneous boundary condition at $x = 0$, say $y(0) = 0$, is self-adjoint (and so enjoys a well-defined spectral theory) without the need to impose any boundary condition at ∞ . In the limit-circle case, on the other hand, the operator does not become

self-adjoint until its domain is restricted by the imposition of some suitable boundary condition at ∞ , and for each of these boundary conditions there is a different spectrum. From the quantum-mechanical point of view, where we expect a well-defined spectrum without the need to impose additional boundary conditions, the limit-point case is the more natural, but a discussion of this and some related topics is given in [10].

If we turn now to the particular case of (1.2), one simple remark can be made at the outset and this is that, if $\alpha \leq 2$, (1.2) makes (1.1) limit-point for all β . This follows from the well-known Levinson limit-point criterion $q(x) \geq -kx^2$ [4, p. 231], k a positive constant, which is applicable if $\alpha \leq 2$ because $p(t)$, being periodic, is bounded below. The situation is less simple if $\alpha > 2$ and the object of this paper is to analyse the limit-point, limit-circle nature of (1.2) for all α and β . In view of the simple remark made above, we assume from now on that $\alpha > 2$.

A partial analysis of two particular cases of (1.2) has already appeared in the literature. The first case is $p(t) = \sin t$, for which (1.2) was shown by Eastham [5] (see also [12]) to be limit-point if $\beta \leq 2$. The range $\beta \leq 1$ had previously been covered by the work of Hartman [11] and McLeod [15-17]. The second case is $p(t) = -1 + k \sin t$, where k is a constant. This time (1.2) was shown by Eastham [6] to be limit-circle if $\beta > \frac{7}{8}\alpha + \frac{5}{4}$ and to be limit-point if $\beta \leq 2$ and $|k| > 1$ (see also [7]). Some corresponding results for fourth-order differential equations have been given recently by Atkinson [2] and Eastham [9].

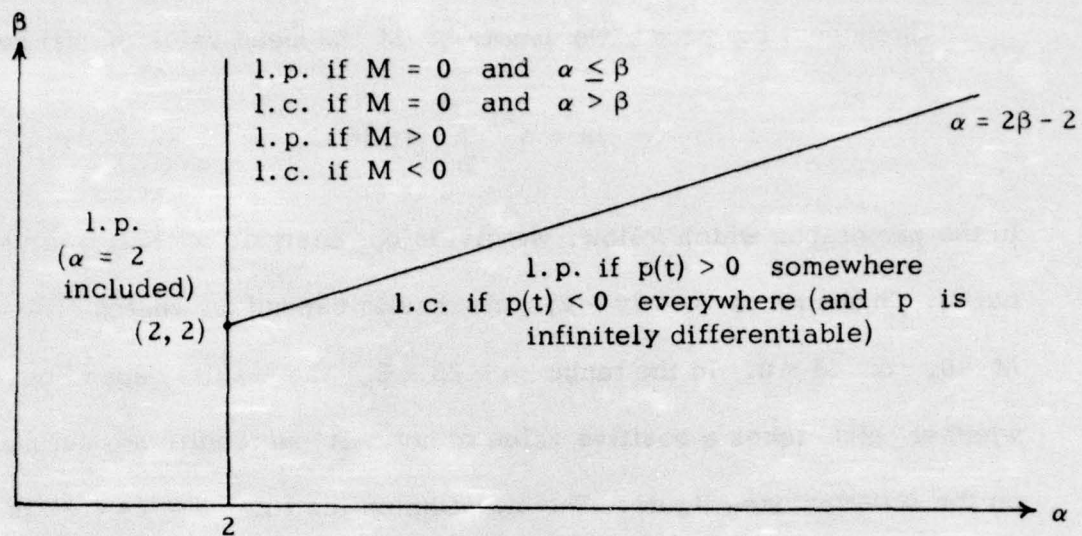
Throughout the paper, we denote by M the mean value of $p(t)$ over $(0, a)$, i.e.

$$M = a^{-1} \int_0^a p(t) dt .$$

In the paragraphs which follow, we divide our analysis of (1.2) into various cases. In the range $\alpha < 2\beta - 2$, the results depend on whether $M = 0$, $M > 0$, or $M < 0$. In the range $\alpha > 2\beta - 2$, the results depend on whether $p(t)$ takes a positive value or not. These results are summarised on the accompanying figure. The situation on the line $\alpha = 2\beta - 2$ is a special one and is described in §9 below. It will be seen from the figure that our analysis is complete as far as the regions $\alpha \leq 2$ and $\alpha < 2\beta - 2$ are concerned. For the region in which $\alpha > 2$ and $\alpha > 2\beta - 2$ our analysis is incomplete in that

(i) when $p(t) < 0$ everywhere, differentiability conditions are imposed on $p(t)$ (see §7 below for a more detailed statement of these conditions);

(ii) the case where $p(t) \leq 0$ but $p(t) \not\equiv 0$ everywhere is not fully dealt with. The situation seems to depend not only on α and β but also on the order of the zeros of $p(t)$. The information that we have on this case is given in §8 below.



2. The case $M = 0$, $\alpha \leq \beta$

We define

$$P(t) = \int_0^t p(u) du .$$

Then the condition $M = 0$ implies that $P(t)$ has period a and hence that $P(t)$ is bounded for all t . We refer now to a particular case of a limit-point criterion of Brinck [3], that (1.1) is limit-point if

$$\int_J x^{-1} q(x) dx \geq -C \quad (2.1)$$

for all intervals J in, say, $[1, \infty)$ of length ≤ 1 , where C is a constant. In our case of (1.2), we have

$$\begin{aligned} \int_J x^{-1} q(x) dx &= \beta^{-1} \int_J x^{\alpha-\beta} p(x^\beta) d(x^\beta) \\ &= \beta^{-1} [x^{\alpha-\beta} P(x^\beta)]_J - \beta^{-1}(\alpha - \beta) \int_J x^{\alpha-\beta-1} P(x^\beta) dx . \end{aligned}$$

Since $P(x^\beta)$ is bounded for all $x \geq 0$ and since we are assuming in this section that $\alpha \leq \beta$, we have

$$\left| \int_J x^{-1} q(x) dx \right| \leq C$$

for some constant C , and so (2.1) is certainly satisfied.

That oscillating potentials of the kind considered here might be covered by (2.1) was suggested by Brinck himself [3, p.229] and he gave the example $q(x) = x^\alpha \sin(x^{\alpha+1})$.

The result, then, of this section is:

A. Let $M = 0$ and let $\alpha \leq \beta$. Then (1.2) makes (1.1) limit-point.

We remark that A can also be proved by means of a limit-point criterion which is of the same kind as the one in [5] and can even be deduced from it - that (1.1) is limit-point if there is a sequence of non-overlapping intervals (a_m, b_m) in $[0, \infty)$ with $\sum (b_m - a_m)^2 = \infty$ and such that

$$(b_m - a_m) \int_J q(x) dx \geq -C \quad (2.2)$$

for all intervals $J \subset (a_m, b_m)$. This criterion is given specifically in [2] as a particular case of results for higher-order differential equations.

It is also of the same nature as the criterion in [3]. The choice to be made in our case of (1.2) is $a_m = m^{\frac{1}{2}}$, $b_m = m^{\frac{1}{2}} + \frac{1}{4} m^{-\frac{1}{2}}$.

3. The case $M = 0$, $\beta < \alpha < 2\beta - 2$

We note that, since $\alpha > 2$, the condition $\alpha < 2\beta - 2$ implies that $\beta > 2$. Hence the stated range $\beta < \alpha < 2\beta - 2$ is meaningful.

We transform (1.1 - 2) to a more manageable form by means of the transformation of Liouville type

$$t = x^\beta, \quad z(t) = x^{(\beta-1)/2} y(x). \quad (3.1)$$

Then we obtain

$$\ddot{z}(t) + \{bt^{-2} - \beta^{-2}t^{-2\gamma}p(t)\}z(t) = 0, \quad (3.2)$$

where $b = \frac{1}{4}(1 - \beta^{-2})$ and

$$2\gamma = 2 - (\alpha + 2)/\beta. \quad (3.3)$$

In this section, we determine the asymptotic form of the solutions of (3.2) as $t \rightarrow \infty$. Our method requires that

$$0 < 2\gamma < 1, \quad (3.4)$$

i.e., by (3.3),

$$\beta - 2 < \alpha < 2\beta - 2, \quad (3.5)$$

and this is certainly ensured by the stated range $\beta < \alpha < 2\beta - 2$.

In (3.2) we substitute

$$z(t) = u(t)v(t), \quad (3.6)$$

where

$$v(t) = t^\gamma \left\{ 1 + \sum_{n=1}^N p_n(t)t^{-2n\gamma} + r(t)t^{-2\gamma-1} \right\}. \quad (3.7)$$

Here the integer N is chosen to make

$$(2N + 1)\gamma > 3\gamma + 1 \quad (3.8)$$

and the $p_n(t)$ and $r(t)$ are twice continuously differentiable periodic functions, with period a , which are defined below.

With the substitution (3.6), (3.2) becomes

$$v^2 \frac{d}{dt} v^2 \frac{du}{dt} + uv^3 [\ddot{v} + \{bt^{-2} - \beta^{-2} t^{-2\gamma} p(t)\}v] = 0. \quad (3.9)$$

Our intention is that the coefficient of u in (3.9) should approach a positive constant as $t \rightarrow \infty$.

Now v^3 has the form

$$v^3 = t^{3\gamma} \left\{ 1 + \sum_{l=1}^{3N} r_n(t) t^{-2n\gamma} + O(t^{-2\gamma-1}) \right\}, \quad (3.10)$$

where $r_n(t)$ has period a , and $r_n(t)$ does not involve $p_{n+1}(t), \dots, p_N(t)$.

In particular, we note that

$$r_1(t) = 3p_1(t). \quad (3.11)$$

Also,

$$\begin{aligned} \ddot{v} + \{bt^{-2} - \beta^{-2} t^{-2\gamma} p(t)\}v = & t^\gamma \sum_{l=1}^N t^{-2n\gamma} \{ \ddot{p}_n(t) - \beta^{-2} p(t) p_{n-1}(t) \} + \{b + \gamma(\gamma-1)\} t^{\gamma-2} \\ & - 2\gamma t^{-\gamma-1} p_1(t) + t^{-\gamma-1} \ddot{r}(t) + O(t^{-3\gamma-1}) + O(t^{-(2N+1)\gamma}), \end{aligned}$$

where $p_0(t) = 1$ and the O -terms refer to $t \rightarrow \infty$. By (3.8), the second O -term can be neglected. Hence, using also (3.10), the coefficient of u in (3.9) has the form

$$t^{3\gamma} (t^\gamma \sum_{l=1}^N t^{-2n\gamma} \{\ddot{p}_n(t) + s_n(t)\} + \{b + \gamma(\gamma - 1)\} t^{\gamma-2} - 2\gamma t^{-\gamma-1} \dot{p}_1(t) + t^{-\gamma-1} \ddot{r}(t) + O(t^{-3\gamma-1})) , \quad (3.12)$$

where $s_n(t)$ involves, besides $p(t)$, at most those $p_j(t)$ and $r_j(t)$ with $j \leq n-1$ and hence at most the $p_j(t)$ with $j \leq n-1$. We note in particular that

$$s_1(t) = -\beta^{-2} p(t) \quad (3.13)$$

and

$$s_2(t) = -\beta^{-2} p(t) p_1(t) + \{\ddot{p}_1(t) - \beta^{-2} p(t)\} r_1(t) . \quad (3.14)$$

Let M_n denote the mean value of $s_n(t)$ over $(0, a)$. Then the periodic functions $p_n(t)$ are defined for $n = 1, 2, \dots, N$ in turn by

$$\ddot{p}_n(t) = -s_n(t) + M_n . \quad (3.15)$$

Also, the periodic function $r(t)$ is defined by

$$\ddot{r}(t) = 2\gamma \dot{p}_1(t) . \quad (3.16)$$

We note that, by (3.13),

$$M_1 = -\beta^{-2} M = 0 . \quad (3.17)$$

Also, since $\ddot{p}_1(t) = -s_1(t) = \beta^{-2} p(t)$, again by (3.13), (3.14) gives

$$s_2(t) = -p_1(t) \ddot{p}_1(t) .$$

Hence

$$aM_2 = \int_0^a s_2(t) dt = -[p_1(t) \dot{p}_1(t)]_0^a + \int_0^a \dot{p}_1^2(t) dt .$$

Thus $M_2 > 0$ and we write $M_2 = A^2$. We now substitute (3.15) and (3.16) into (3.12). Then (3.12) takes the form

$$A^2 + R(t) + \{b + \gamma(\gamma - 1)\}t^{4\gamma-2} + O(t^{-1}), \quad (3.18)$$

where

$$R(t) = \sum_3^N M_n t^{-(2n-4)\gamma}. \quad (3.19)$$

In (3.9), we now make the change of variable

$$\xi = \int_0^t v^{-2}(t) dt = (1 - 2\gamma)^{-1} t^{1-2\gamma} \{1 + O(t^{-2\gamma})\}, \quad (3.20)$$

except that the O -term would be $O(t^{-\frac{1}{2}} \log t)$ if $2\gamma = \frac{1}{2}$. Then, writing

$$u(t) = U(\xi), \quad R(t) = R_1(\xi)$$

and using (3.18), we can write (3.9) as

$$\frac{d^2 U}{d\xi^2} + U\{A^2 + R_1(\xi) + O(\xi^{-d})\} = 0, \quad (3.21)$$

where

$$d = \min\{2, (1 - 2\gamma)^{-1}\} > 1. \quad (3.22)$$

We note that, by (3.4) and (3.20), $\xi \rightarrow \infty$ as $t \rightarrow \infty$.

By (3.19), we have

$$\int_0^\infty \left| \frac{dR_1(\xi)}{d\xi} \right| d\xi = \int_0^\infty \left| \frac{dR(t)}{dt} \right| dt < \infty,$$

and hence the asymptotic form of the solutions of (3.21) as $\xi \rightarrow \infty$ follows

from the remarks on pp. 91-92 of [4]. Thus (3.21) has two solutions

which are asymptotic respectively to

$$\exp(\pm i \int_0^{\xi} \{A^2 + R_1(\xi)\}^{\frac{1}{2}} d\xi) .$$

In particular, going back through (3.7), (3.6), and (3.1), we find that

(1.1) has two solutions $y_j(x)$, ($j = 1, 2$) such that

$$|y_j(x)| \sim x^{\beta\gamma - (\beta-1)/2} ,$$

i.e., by (3.3),

$$|y_j(x)| \sim x^{(\beta-\alpha-1)/2} , \quad (3.23)$$

as $x \rightarrow \infty$. If $\alpha > \beta$, these two solutions are both $L^2(0, \infty)$, and so we have the limit-circle case for (1.1). Thus the result of this section is:

B. Let $M = 0$ and let $\beta < \alpha < 2\beta - 2$. Then (1.2) makes (1.1) limit-circle.

We pointed out in (3.4 - 5) that the above method up to (3.23) works when $\beta - 2 < \alpha < 2\beta - 2$. Therefore it also follows from (3.23) that, when $\beta - 2 < \alpha \leq \beta$, (1.2) makes (1.1) limit-point and, to this extent, we have an overlap with the result A of §2.

We conclude this section by mentioning, first, that our method has some points of similarity with the one indicated in §§2 and 5 of [18] and, secondly, that a possible alternative method would be to compare (3.2) with the periodic equation $\ddot{z}(t) - \epsilon p(t)z(t) = 0$, where ϵ is a small parameter (cf. [1]).

4. The case $M > 0, \alpha < 2\beta - 2$

Suppose first that $\alpha \leq \beta$. Then we can use (2.1) again because now

$$\begin{aligned} \int_J x^{-1} q(x) dx &= \int_J M x^{\alpha-1} dx + \int_J x^{-1+\alpha} \{p(x^\beta) - M\} dx \\ &\geq \int_J x^{-1+\alpha} \{p(x^\beta) - M\} dx, \end{aligned}$$

and this integral is bounded as in §2 since $p(t) - M$ has mean value zero. Thus the limit-point case occurs.

Now suppose that $\beta < \alpha < 2\beta - 2$. Here we need only take the case $N = 1$ of (3.7) and omit the term involving $r(t)$. Thus we define

$$v(t) = t^\gamma \{1 + p_1(t) t^{-2\gamma}\},$$

where $p_1(t)$ is the periodic function defined by

$$\ddot{p}_1(t) = \beta^{-2} \{p(t) - M\}. \quad (4.1)$$

Then

$$\begin{aligned} \ddot{v} + \{bt^{-2} - \beta^{-2} t^{-2\gamma} p(t)\} v \\ = \gamma(\gamma-1)t^{\gamma-2} + \beta^{-2} t^{-\gamma} \{p(t) - M\} + bt^{\gamma-2} - \beta^{-2} t^{-\gamma} p(t) + o(t^{-\gamma}) = -M\beta^{-2} t^{-\gamma} + o(t^{-\gamma}) \end{aligned}$$

since $\gamma - 2 < -\gamma$. Hence (3.9) is

$$v^2 \frac{d}{dt} v^2 \frac{du}{dt} - u \{M\beta^{-2} t^{2\gamma} + o(t^{2\gamma})\} = 0.$$

Since $\{\dots\}$ here is large and positive for large t , this equation has solutions which are exponentially (and more) large in $\xi = \int_0^t dt/v^2$. Thus

we have a solution $u(t)$ such that $u(t) > \exp(k t^{1-2\gamma})$, where $k > 0$.

Then certainly the corresponding solution $y(x)$ of (1.1) is not $L^2(0, \infty)$,

and again the limit-point case occurs. Thus the result of this section is:

C. Let $M > 0$ and let $\alpha < 2\beta - 2$. Then (1.2) makes (1.1) limit-point.

5. The case $M < 0$, $\alpha < 2\beta - 2$

Suppose first that $\beta > \frac{7}{8}\alpha + \frac{5}{4}$. Then the situation is covered by the analysis in §§3 - 4 of [6] - see especially (4.4) and (4.5) of [6].

We again define $p_1(t)$ by (4.1) above and then define

$$s(x) = x^{\alpha-2\beta+2} p_1(x^\beta).$$

Hence (1.2) can be written

$$q(x) = Mx^\alpha + s''(x) + O(x^{\alpha-\beta}).$$

and $s(x)$ satisfies the conditions on S in Theorem 2 (and its modification) in [6]. Then, as in [6], we have the limit-circle case.

Suppose next that $\beta - 2 < \alpha < 2\beta - 2$. Thus (3.5) holds and we shall consider again the method of §3. We note first, however, that these two sub-cases $\beta > \frac{7}{8}\alpha + \frac{5}{4}$ and $\beta - 2 < \alpha < 2\beta - 2$ overlap and between them make up the whole of $\alpha < 2\beta - 2$, subject of course to the condition $\alpha > 2$ which is assumed throughout.

We make the substitution (3.6 - 7) again. The condition $M = 0$ which was imposed in §3 was not in fact used until (3.17). If now $M \neq 0$, (3.18) is replaced by

$$S(t) + \{b + \gamma(\gamma - 1)\}t^{4\gamma-2} + O(t^{-1}),$$

where

$$S(t) = \sum_{n=1}^N M_n t^{-(2n-4)\gamma}$$

and, as in (3.17), $M_1 = -\beta^{-2}M (\neq 0 \text{ now})$. Thus the leading term in $S(t)$ is

$$-\beta^{-2}Mt^{2\gamma}, \quad (5.1)$$

which is large and positive as $t \rightarrow \infty$. Correspondingly, (3.21) is replaced by

$$\frac{d^2 U}{d\xi^2} + U\{S_1(\xi) + O(\xi^{-d})\} = 0,$$

where $S_1(\xi) = S(t)$.

We now substitute

$$U(\xi) = S_1^{-1/4}(\xi)w(\xi) \quad (5.2)$$

and write

$$\eta = \int^\xi S_1^{1/2}(\xi) d\xi. \quad (5.3)$$

Then we obtain, as we did (3.9) and (3.21),

$$\frac{d^2 W}{d\eta^2} + W(1 + S_1^{-3/4}(\xi) \frac{d^2}{d\xi^2} S_1^{-1/4}(\xi) + O\{\xi^{-d} S_1^{-1}(\xi)\}) = 0, \quad (5.4)$$

where $W(\eta) = w(\xi)$. By (5.1) and (3.20), the coefficient of W here is

$$1 + O(t^{-1}) + O(t^{-d(1-2\gamma)-2\gamma}). \quad (5.5)$$

Since $2\gamma < 1$ and $d > 1$, by (3.4) and (3.22), we have

$$-d(1-2\gamma) - 2\gamma < -1,$$

this inequality being a re-arrangement of $(d-1)(1-2\gamma) > 0$. Hence

(5.5) is

$$1 + O(t^{-1}) = 1 + O(\eta^{-1/(1-\gamma)})$$

by (5.3) and (3.20). Since $1/(1-\gamma) > 1$, we can again quote pp. 91 - 92 of [4] to say that all solutions $W(\eta)$ of (5.4) are bounded as $\eta \rightarrow \infty$. Hence, going back through (5.2), (5.1), (3.7), (3.6), and (3.1), we find that all solutions $y(x)$ of (1.1) are

$$O(x^{\{\beta\gamma-(\beta-1)\}/2}) = O(x^{-\alpha/4})$$

as $x \rightarrow \infty$. Since $\alpha > 2$, all solutions of (1.1) are, therefore, $L^2(0, \infty)$ and we have the limit-circle case. Thus the result of this section is:

D. Let $M < 0$ and let $\alpha < 2\beta - 2$. Then (1.2) makes (1.1) limit-circle.

6. The case $\alpha > 2\beta - 2$, $p(t)$ taking positive values

In (3.3), we now have $\gamma < 0$ and we write $\gamma = -\delta$, so that $\delta > 0$. Then in (3.2) we write

$$F(t) = \beta^{-2} t^{2\delta} p(t) - bt^{-2},$$

so that

$$\ddot{z}(t) = F(t)z(t). \quad (6.1)$$

Since $p(t)$ is a periodic function which is now assumed to take positive values, we can say that there are positive constants $A^2, B^2, \theta_1, \theta_2$ such that

$$A^2 t^{2\delta} \leq F(t) \leq B^2 t^{2\delta} \quad (6.2)$$

in the intervals $(\theta_1 + na, \theta_2 + na)$, $n = 0, 1, \dots$. We call these intervals (a_n, b_n) . By taking $\theta_2 - \theta_1$ small enough, we can arrange that

$$B < \frac{3}{2} A. \quad (6.3)$$

Let $z_{n,1}(t)$ and $z_{n,2}(t)$ be the solutions of (6.1) defined by

$$z_{n,1}(a_n) = 0, \quad \dot{z}_{n,1}(a_n) = 1, \quad (6.4)$$

$$z_{n,2}(t) = z_{n,1}(t) \int_t^{b_n} z_{n,1}^{-2}(u) du. \quad (6.5)$$

We note that

$$W(z_{n,1}, z_{n,2})(t) = -1. \quad (6.6)$$

Let N be any integer. From (6.2), we have for $n \geq N$

$$\mu_n^2 \leq F(t) \leq K^2 \mu_n^2 \quad (6.7)$$

in (a_n, b_n) , where

$$\mu_n = A a_n^\delta \quad \text{and} \quad K = (B/A) \sup_{n \geq N} (b_n/a_n)^\delta.$$

By (6.3), we can choose N so that

$$K < \frac{3}{2}. \quad (6.8)$$

By (6.7), the theory of differential inequalities [20, p. 69] applied to (6.1) and (6.4) gives

$$\mu_n^{-1} \sinh\{\mu_n(t - a_n)\} \leq z_{n,1}(t) \leq (K\mu_n)^{-1} \sinh\{K\mu_n(t - a_n)\}. \quad (6.9)$$

Then (6.5) gives

$$z_{n,2}(t) \leq \frac{\sinh\{K\mu_n(t - a_n)\} \sinh\{\mu_n(b_n - t)\}}{K \sinh\{\mu_n(t - a_n)\} \sinh\{\mu_n(b_n - a_n)\}} \quad (6.10)$$

and

$$z_{n,2}(t) \geq \frac{K \sinh\{\mu_n(t - a_n)\} \sinh\{K\mu_n(b_n - t)\}}{\sinh\{K\mu_n(t - a_n)\} \sinh\{K\mu_n(b_n - a_n)\}}. \quad (6.11)$$

Now consider any two real solutions $z_1(t)$, $z_2(t)$ of (6.1) such that

$$W(z_1, z_2)(t) = 1. \quad (6.12)$$

In (a_n, b_n) , we must have

$$z_1(t) = A_n z_{n,1}(t) + B_n z_{n,2}(t),$$

$$z_2(t) = C_n z_{n,1}(t) + D_n z_{n,2}(t),$$

and (6.6) and (6.12) imply that

$$A_n D_n - B_n C_n = -1. \quad (6.13)$$

It follows from (3.1) that (1.1) will be limit-point if, for all intervals

(a_n, b_n) with n large enough,

$$\text{either } \int_{a_n}^{b_n} z_1^2(t) t^{-2+2/\beta} dt \geq k \quad \text{or} \quad \int_{a_n}^{b_n} z_2^2(t) t^{-2+2/\beta} dt \geq k, \quad (6.14)$$

k being a positive constant independent of n . Now,

$$\int_{a_n}^{b_n} z_1^2(t) t^{-2+2/\beta} dt = A_{nn,1}^2 I_{nn,1} + 2A_{nn} B_{nn} J_n + B_{nn,2}^2 I_{nn,2}, \quad (6.15)$$

where

$$\begin{aligned} I_{n,1} &= \int_{a_n}^{b_n} z_{n,1}^2(t) t^{-2+2/\beta} dt \geq k \mu_n^{-3} a_n^{-2+2/\beta} e^{2\mu_n(b_n - a_n)}, \\ J_n &= \int_{a_n}^{b_n} z_{n,1}(t) z_{n,2}(t) t^{-2+2/\beta} dt \leq k \mu_n^{-2} a_n^{-2+2/\beta} e^{2(K-1)\mu_n(b_n - a_n)}, \\ I_{n,2} &= \int_{a_n}^{b_n} z_{n,2}^2(t) t^{-2+2/\beta} dt \geq k \mu_n^{-1} a_n^{-2+2/\beta}, \end{aligned}$$

on using (6.9), (6.10), and (6.11). By completing the square in (6.15),

we see that

$$\int_{a_n}^{b_n} z_1^2(t) t^{-2+2/\beta} dt \geq A_n^2 (I_{n,1} I_{n,2} - J_n^2) / I_{n,2}.$$

Then (6.14) certainly follows for $z_1(t)$ if

$$A_n^2 \geq k \mu_n^{-3} a_n^{-2+2/\beta} e^{-2\mu_n(b_n - a_n)}, \quad (6.16)$$

where, in neglecting J_n^2 in comparison to $I_{n,1}I_{n,2}$, we have used the inequality $4(K-1) < 2$, which is implied by (6.8). Similarly, (6.14) follows for $z_2(t)$ if

$$C_n^2 \geq k\mu_n^3 a_n^{2-2/\beta} e^{-2\mu_n(b_n - a_n)}. \quad (6.17)$$

If neither (6.16) nor (6.17) holds, then, by (6.13), we must have either

$$B_n^2 \geq k\mu_n^{-3} a_n^{-2+2/\beta} e^{2\mu_n(b_n - a_n)} \quad (6.18)$$

or

$$D_n^2 \geq k\mu_n^{-3} a_n^{-2+2/\beta} e^{2\mu_n(b_n - a_n)}, \quad (6.19)$$

and then the inequality

$$\int_{a_n}^{b_n} z_1^2(t) t^{-2+2/\beta} dt \geq B_n^2 (I_{n,1}I_{n,2} - J_n^2) / I_{n,1},$$

obtained again from (6.15) by completing the square, gives (6.14) for $z_1(t)$ in the case of (6.18). We can argue similarly for $z_2(t)$ in the case of (6.19). Hence the result of this section is:

E. Let $\alpha > 2\beta - 2$ and let $p(t)$ take positive values. Then (1.2) makes (1.1) limit-point.

We remark that the result E for the more restricted range $\alpha \geq 4\beta - 6$ follows from a general limit-point criterion of Ismagilov [13] - see also [14].

7. The case $\alpha > 2\beta - 2$, $p(t) < 0$ for all t

As in §6, we write $\gamma = -\delta$ in (3.2) and, since $p(t) < 0$ now, we can write $\beta^{-2}p(t) = -Q^{-4}(t)$, where $Q(t) > 0$. Then (3.2) is

$$\ddot{z}(t) + \{bt^{-2} + t^{2\delta}Q^{-4}(t)\}z(t) = 0. \quad (7.1)$$

We make the substitution (3.6) again:

$$z(t) = u(t)v(t) \quad (7.2)$$

but, instead of (3.7), we take

$$v(t) = t^{-\frac{1}{2}\delta}Q(t)\{1 + \sum_1^N p_n(t)t^{-2n\delta}\}, \quad (7.3)$$

where the integer N is chosen to make

$$2(N+1)\delta > \delta + 1 \quad (7.4)$$

and the $p_n(t)$, to be defined below, have period a . Then, as for (3.9),

we have

$$\begin{aligned} & v^2 \frac{d}{dt} v^2 \frac{du}{dt} + u \left[t^{-\frac{3}{2}\delta} Q^3(t) \{1 + \sum_1^N p_n(t)t^{-2n\delta}\}^3 \times \right. \\ & \quad \times \left(t^{-\frac{1}{2}\delta} \dot{Q}(t) \{1 + \sum_1^N p_n(t)t^{-2n\delta}\} + 2t^{-\frac{1}{2}\delta} \dot{Q}(t) \sum_1^N \dot{p}_n(t)t^{-2n\delta} + \right. \\ & \quad \left. \left. + t^{-\frac{1}{2}\delta} Q(t) \sum_1^N \ddot{p}_n(t)t^{-2n\delta} + O(t^{-\frac{1}{2}\delta-1}) \right) \right] + \\ & \quad \left. + \{bt^{-2} + t^{2\delta}Q^{-4}(t)\}t^{-2\delta}Q^4(t)\{1 + \sum_1^N p_n(t)t^{-2n\delta}\}^4 \right] = 0, \end{aligned}$$

which is

$$\begin{aligned}
 v^2 \frac{d}{dt} v^2 \frac{du}{dt} + u \left[t^{-2\delta} Q^3(t) \left\{ 1 + \sum_{n=1}^N p_n(t) t^{-2n\delta} \right\}^3 \times \right. \\
 \times \left(\ddot{Q}(t) \left\{ 1 + \sum_{n=1}^N p_n(t) t^{-2n\delta} \right\} + \right. \\
 \left. + 2\dot{Q}(t) \sum_{n=1}^N \dot{p}_n(t) t^{-2n\delta} + Q(t) \sum_{n=1}^N \ddot{p}_n(t) t^{-2n\delta} \right) + \\
 \left. + \left\{ 1 + \sum_{n=1}^N p_n(t) t^{-2n\delta} \right\}^4 + O(t^{-2\delta-1}) \right] = 0. \quad (7.5)
 \end{aligned}$$

On expanding $\{\dots\}^4$ here by the binomial theorem, we obtain (inter alia) the terms $4p_n(t)t^{-2n\delta}$ ($n = 1, \dots, N$). Then $p_n(t)$ is chosen so that there is no term involving $t^{-2n\delta}$ in $[\dots]$. Thus

$$p_1(t) = -\frac{1}{4} Q^3(t) \ddot{Q}(t)$$

and $p_n(t)$ ($n \geq 2$) involves $p_1(t), \dots, p_{n-1}(t)$. Hence the $p_n(t)$ are determined in turn.

It is clear also that $p_N(t)$ involves $Q^{(2N)}(t)$ and so we must assume the existence and continuity of $Q^{(2N+2)}(t)$ and hence of $p^{(2N+2)}(t)$. The nearer δ is to zero, the larger N is (by (7.4)) and the greater the differentiability required of $p(t)$. If $\delta > 1$ (for example), i.e. if $\alpha > 4\beta - 2$, we can take $N = 0$ in (7.4) and then we need only assume the existence and continuity of $\ddot{p}(t)$. More generally, if

$p^{(2N+2)}(t)$ exists and is continuous, we can deal with the region $\alpha > 2\beta \frac{2N+2}{2N+1} - 2$ since this last inequality is just a re-arrangement of (7.4). If, therefore, we wish to deal with the entire region $\alpha > 2\beta - 2$ with a single differentiability condition on $p(t)$, that condition has to be that $p(t)$ is infinitely differentiable.

With our choice of the $p_n(t)$ described above, (7.5) takes the form

$$v^2 \frac{d}{dt} v^2 \frac{du}{dt} + u[1 + O(t^{-2(N+1)\delta}) + O(t^{-2\delta-1})] = 0. \quad (7.6)$$

We make the change of variable

$$\xi = \int_0^t v^{-2}(t) dt.$$

By (7.3), $\xi/t^{\delta+1}$ lies between positive constants as $t \rightarrow \infty$. Hence (7.6) becomes

$$\frac{d^2 U}{d\xi^2} + U\{1 + O(\xi^{-d})\} = 0, \quad (7.7)$$

where $U(\xi) = u(t)$, $d > 1$, and we have used (7.4).

Since all solutions $U(\xi)$ of (7.7) are bounded as $\xi \rightarrow \infty$, again by pp. 91 - 92 of [4], it follows from (7.2) and (7.3) that all solutions $z(t)$ of (3.2) are $O(t^{-\frac{1}{2}\delta})$ as $t \rightarrow \infty$. Hence, by (3.1), all solutions $y(x)$ of (1.1) are $O(x^{-\frac{1}{2}\beta\delta - (\beta-1)/2}) = O(x^{-\frac{1}{4}\alpha})$ as $x \rightarrow \infty$. Since $\alpha > 2$, all solutions of (1.1) are, therefore, $L^2(0, \infty)$ and we have the limit-circle case. Thus the result of this section is:

F. Let $\alpha > 2\beta - 2$ and let $p(t) < 0$ for all t . Also, let $p(t)$ be infinitely differentiable. Then (1.2) makes (1.1) limit-circle.

8. The case $\alpha > 2\beta - 2$, $p(t) \leq 0$ for all t , and $p(t)$ taking the
value zero

We do not have a complete analysis of this case but we can say enough to indicate that the situation is more complicated than in previous cases in that the order of the zeros of $p(t)$, as well as α and β , appears to affect the limit-point, limit-circle nature of (1.2). We shall give the discussion for the particular potential

$$q(x) = -x^\alpha \sin^{2n}(x^\beta), \quad (8.1)$$

where n is a positive integer, but the ideas require only obvious modifications for suitable more general potentials (1.2). However, it does remain an open question to what extent (8.1) is typical of all potentials (1.2) falling under the heading of this section. There are certainly complications if $p(t)$ has an infinity of zeros of order $2n$ in $(0, a)$, or more generally if it vanishes at a point which is not a zero of a specific order.

We obtain first a limit-point result for (8.1). We take $a_m = (m\pi)^{1/\beta} - m^{-1/2}$ and $b_m = (m\pi)^{1/\beta} + m^{-1/2}$ in (2.2) (or in Corollary 1 of [5]). Then (8.1) will be limit-point if

$$x^\alpha \sin^{2n}(x^\beta) \leq Cm \quad (8.2)$$

in (a_m, b_m) , where C is a constant. To ensure that the (a_m, b_m) are non-overlapping, at any rate when m is large enough, we take

$$\beta < 2. \quad (8.3)$$

Since $x = (m\pi)^{1/\beta} + O(m^{-1/2})$ in (a_m, b_m) , (8.2) is satisfied if

$$\frac{\alpha}{\beta} + 2n\left(\frac{1}{2} - \frac{1}{\beta}\right) \leq 1,$$

i.e. if

$$\alpha + (n-1)\beta \leq 2n.$$

This condition implies (8.3) since $\alpha > 2$. Hence we have

G. Let $\alpha + (n-1)\beta \leq 2n$. Then (8.1) makes (1.1) limit-point.

We now make a conjecture.

H (conjectured). Let $\alpha + (n-1)\beta > 2n$. Then (8.1) makes (1.1) limit-circle.

We support this conjecture with the following remarks. Considering (3.2), we seek an approximation to solutions of

$$\ddot{z}(t) + \{bt^{-2} + \beta^{-2}t^{2\delta} \sin^{2n}t\}z(t) = 0 \quad (8.4)$$

throughout an interval $[(m - \frac{1}{2})\pi, (m + \frac{1}{2})\pi]$, where m is a large integer and, as in §§6 and 7, $\delta = -\gamma > 0$. We define

$$P(t) = bt^{-2} + \beta^{-2}t^{2\delta} \sin^{2n}t,$$

$$\xi(t) = (n+1)^{1/(n+1)} \left(\int_{m\pi}^t P^{\frac{1}{2}}(u) du \right)^{1/(n+1)}, \quad (8.5)$$

and

$$f(t) = \xi^{\frac{1}{2}n}(t) P^{-\frac{1}{4}}(t).$$

Then it can be shown that, both when $t - m\pi$ is small and when $t - m\pi$ is exactly of order 1,

$$(a) \quad f(t) \asymp m^{-\frac{1}{2}\delta/(n+1)}, \quad (8.6)$$

(b) $f(t)W(\xi)$ satisfies an equation approximating to (8.4), where $W(\xi)$ is a solution of

$$\frac{d^2 W}{d\xi^2} + \xi^{2n} W = 0. \quad (8.7)$$

We omit the routine details of the calculations. Now (b) suggests that solutions $z(t)$ of (8.4) are approximately of the form $f(t)W(\xi)$ and hence, by (3.1), that (1.1) is limit-circle if

$$\sum_m \int_{(m-\frac{1}{2})\pi}^{(m+\frac{1}{2})\pi} |f(t)W(\xi)|^2 t^{-2+2/\beta} dt < \infty. \quad (8.8)$$

By (8.5) and (8.6),

$$\frac{d\xi}{dt} = \xi^{-n} P^{\frac{1}{2}}(t) = f^{-2}(t) \asymp m^{\delta/(n+1)}.$$

Hence (8.8) holds if

$$\sum_m m^{-2\delta/(n+1)-2+2/\beta} \int_{\xi\{(m-\frac{1}{2})\pi\}}^{\xi\{(m+\frac{1}{2})\pi\}} |W(\xi)|^2 d\xi < \infty,$$

where we have used (8.6) again. Since all solutions $W(\xi)$ of (8.7) are $O(\xi^{-\frac{1}{2}n})$ as $\xi \rightarrow \infty$, the integral term here is bounded if $n > 1$. (The case $n = 1$ introduces a negligible logarithm.) Hence (8.8) holds if

$$\frac{2\delta}{n+1} + 2 - \frac{2}{\beta} > 1.$$

Since $2\delta = -2\gamma = -2 + (\alpha + 2)/\beta$, this reduces to $\alpha + (n-1)\beta > 2n$ as required.

The rigorization of this argument would appear to involve some complicated analysis on the lines of the Langer-Titchmarsh approach to turning points. Although it is hoped that a treatment of this will appear in due course, the details have not been carried through at the present time.

9. The case $\alpha = 2\beta - 2$

In this section, it is convenient to consider, in place of (1.1), the equation

$$y''(x) - \{q(x) + b\beta^2 x^{-2}\}y(x) = 0, \quad (9.1)$$

where b is as in (3.2). Since the coefficients of $y(x)$ in (9.1) and (1.1) differ only by a term $b\beta^2 x^{-2}$ which is bounded in the neighborhood of $x = \infty$, the limit-point, limit-circle nature of (1.1) is the same as that of (9.1) at $x = \infty$ (cf. [4, p. 225]). When the transformation (3.1) is applied to (9.1), we obtain, in place of (3.2), the simpler equation

$$\ddot{z}(t) - \beta^{-2} t^{-2\gamma} p(t)z(t) = 0.$$

In the present case when $\alpha = 2\beta - 2$, $\gamma = 0$ and we have the periodic equation

$$\ddot{z}(t) - \beta^{-2} p(t)z(t) = 0. \quad (9.2)$$

The limit-point, limit-circle nature of (1.1) is connected to the stability nature of (9.2), a connection which was noted by Sears [19] in a not dissimilar context. For completeness, we give here the details of this connection and we refer to [8, §§1.1 - 3] for the necessary theory of (9.2).

If (9.2) is stable, all solutions of (9.2) are bounded in $(0, \infty)$ and so, by (3.1), all solutions $y(x)$ of (9.1) are

$$O(x^{-(\beta-1)/2}) = O(x^{-\alpha/4})$$

as $x \rightarrow \infty$. Since $\alpha > 2$, we have the limit-circle case.

If (9.2) is unstable, (9.2) has an exponentially large solution as $t \rightarrow \infty$ and the corresponding $y(x)$ is certainly not $L^2(0, \infty)$. Hence we have the limit-point case.

If (9.2) is conditionally stable, but not stable (i.e. case D2 or case E2 of [8, §1.2] holds), (9.2) has a solution $z(t)$ of the form

$$z(t) = tP_1(t) + P_2(t),$$

where $P_1(t)$ and $P_2(t)$ have period a or $2a$. For this $z(t)$ we have

$$\int_0^\infty |z^2(t)| t^{-2+2/\beta} dt = \infty$$

and hence, by (3.1), the corresponding $y(x)$ is not $L^2(0, \infty)$. Thus we have the limit-point case again.

The result, then, of this section is:

I. Let $\alpha = 2\beta - 2$. If (9.2) is stable, then (1.2) makes (1.1) limit-circle. Otherwise, (1.2) makes (1.1) limit-point.

An example in which both the possibilities in I are realised is that in which (9.2) is the Mathieu equation. Here,

$$p(t) = -\beta^2(\lambda - q \cos 2t),$$

where λ and q are constants with $q \neq 0$. Given q , (9.2) can be made both stable and unstable by choice of λ - see, e.g., [8, §2.5].

Acknowledgement. Work on this paper was done while we were at the University of Dundee during the Summer Term of 1974 in connection with the Symposium on Spectral Theory and Differential Equations, which was supported by the Science Research Council. We thank Professor W. N. Everitt for the invitation to visit the Mathematics Department of the University of Dundee.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 1676	2. GOVT ACCESSION NO.	3. REPORT'S CATALOG NUMBER 9 Technical
4. TITLE (and Subtitle) 6 THE LIMIT-POINT, LIMIT-CIRCLE NATURE OF RAPIDLY OSCILLATING POTENTIALS.		5. TYPE OF REPORT & PERIOD COVERED Summary Report, no specific reporting period
7. AUTHOR(s) 10 F. V. Atkinson, M. S. P. Eastham, and J. B. McLeod		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE 11 September 76
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 14 MRC-TSR-1676		13. NUMBER OF PAGES 30
15. SECURITY CLASS. (of this report) UNCLASSIFIED		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Ordinary differential operators Boundary-value problems Self-adjointness Asymptotics		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The report analyses the Weyl limit-point, limit-circle classification, L^2 square i.e. the number of linearly independent solutions in $L^2(0, \infty)$, of the equation $y''(x) - q(x)y(x) = 0$ ($0 \leq x < \infty$), ∞ where $q(x)$ has the form $q(x) = x^\alpha p(x^\beta)$, x to the alpha power α and β being positive constants and $p(t)$ a real continuous periodic function of t .		

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